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Voting under Constraints*

Salvador Barberà and Jordi Massó

Departament d'Economia i d'Història Econòmica and CODE, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Spain

and

Alejandro Neme

Instituto de Matemática Aplicada, Universidad Nacional de San Luis, Ejército de los Andes 950, 5700 San Luis, Argentina

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We consider a broad class of situations where a society must choose from a finite set of alternatives. This class includes, as polar cases, those where the preferences of agents are completely unrestricted and those where their preferences are single-peaked. We prove that strategy-proof mechanisms in all these domains must be based on a generalization of the median voter principle. Moreover, they must satisfy a property, to be called the "intersection property," which becomes increasingly stringent as the preference domain is enlarged. In most applications, our results precipitate impossibility theorems. In particular, they imply the Gibbard–Satterthwaite theorem as a corollary. *Journal of Economic Literature* Classification Number: D71. © 1997 Academic Press

1. INTRODUCTION

We consider a broad class of situations where a society must choose from a finite set of alternatives, and we provide a full characterization of the set of strategy-proof social choice functions under such situations. Our class includes, along with many others, the polar cases where the preferences of agents are completely unrestricted and where these preferences are

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single-peaked in one dimension. Our general characterization implies, as corollaries, the impossibility theorem of Gibbard and Satterthwaite, as well as Moulin's characterization of strategy-proof voting schemes on the line. Hence, results that have been widely perceived as very different appear here as the common consequence of some fundamental facts. These are, first, that strategy-proof mechanisms must be based on an appropriate generalization of the median voter rule and, second, that they must satisfy the intersection property.

This condition, to be described below, becomes increasingly stringent and forces dictatorship as the preference domain is enlarged. Consequently positive results only arise under very strong domain restrictions. On balance our results confirm the widespread pessimism that exists about the possibility of designing strategy-proof mechanisms.

A society must choose from a finite set of alternatives, and each alternative α can be described as a K-tuple of integer values $(\alpha_1, ..., \alpha_K)$, with each α_k belonging to a pre-specified integer interval $[a_k, b_k]$. Each dimension k stands for one possible relevant characteristic of our alternatives, the elements in the integer interval $[a_k, b_k]$ describe the levels at which the kth characteristic may be satisfied, and alternatives can be identified by the levels $(\alpha_1, ..., \alpha_K)$ at which they perform on each of the characteristics. For example, a firm may have to choose sets of employees from a given set of applicants $\mathbb{K} = \{1, 2, ..., K\}$. Since each subset of $\mathbb{K}$ may be described by its characteristic function, subsets of applicants can be identified with K-tuples in $B = \prod_{k=1}^{K} \{0, 1\}$, with $\alpha_k = 0$ if the kth applicant is not chosen and $\alpha_k = 1$ if he is. Since modeling alternatives for collective choice as multidimensional vectors has a long tradition in economics and political science, the reader will find many other situations where to apply this basic framework. Yet, our analysis does not cover all types of multidimensional alternatives. Consistency with some of the assumptions to come make it best to consider interpretations where a higher level of the characteristic is not necessarily associated to higher satisfaction for the voter. Locational characteristics are a good example of cases we cover. Amounts spent on public goods are not, but other qualitative features of a public project may well be encompassed.

What is specific to our paper is the incorporation of different types of constraints to collective decision-making. The firm in our example may have a limited number of vacancies, say three. Some of these vacancies, say one, may have to be filled necessarily, while the others may or may not, depending on the quality of applicants. Thus, although any subset of applicants is a potential or conceivable alternative, only subsets with at least one applicant and at most three are feasible. Thus, alternatives will typically belong to a Cartesian product (also called a box), but the range of feasible alternatives will typically *not* be a Cartesian product itself.

We investigate the set of strategy-proof social choice functions selecting feasible alternatives, when the preferences of voters satisfy a generalized version of single-peakedness. When all K-tuples of characteristics are feasible, this set is known to coincide with the family of all generalized median voter schemes (Barberà, Gul and Stacchetti [2]). We prove (Theorem 3) that these methods are still the only possible candidates for strategy-proof decision making in our setting. Those are based on a strong form of decentralized decision-making: a value is selected for each of the characteristics describing alternatives, and then the alternative jointly defined by these independently selected values is the recommended social outcome. To what extent can we still use generalized median voter schemes to make social choices when feasibility restrictions preclude some K-tuples of values as possible outcomes?. Of course, if we allow agents to vote for unfeasible alternatives in the range of a given generalized median voter scheme f, then unfeasible recommendations will emerge. Moreover, even if agents are restricted to vote for feasible alternatives, some combinations of individually admissible votes may result in unfeasible collective recommendations. Whether or not this problem arises depends jointly on the shape of the feasible set and on the specific generalized median voter scheme under consideration. We prove (Theorem 1) that generalized median voter schemes respect feasibility if and only if they satisfy the intersection property. The set of such generalized median voter schemes constitutes the class of all strategy-proof social choice functions under the strong assumption that the domain of preferences only includes single-peaked preferences whose top alternative is feasible. The intersection property is automatically satisfied if the set of feasible outcomes is Cartesian, but otherwise becomes very stringent. For many shapes of the feasible set it precipitates impossibility results. In other cases, nontrivial social choice functions satisfy the intersection property. Yet, these positive cases should be qualified, because they may not be robust to relaxations of the strong assumption that the only admissible preferences are those with feasible tops. At any rate, our unified treatment identifies the intersection property as a condition to be met, and provides a systematic approach to discover the structure of strategy-proof social choice functions under any type of constraints.

The role of single-peakedness in avoiding strategic manipulations is already mentioned in Black's [5] seminal article. Moulin [12] proved that generalized median voter schemes on the real line were the only strategy-proof mechanisms among those requiring agents to reveal only their preferred alternative. Barberà and Jackson [4] extended the analysis to cover all closed ranges. In that more general setting, they also proved that, in fact, dropping the "tops only" requirement does not enlarge the set of strategy-proof mechanisms, a fact already proven by Sprumont [15] for the real line.

The first extensions of these results from one to several dimensions are due to Laffond [10], Chichilnisky and Heal [8] and Border and Jordan [6]. The papers by Barberà, Sonnenschein and Zhou [3], Barberà, Gul and Stacchetti [2], Bossert and Weymark [7], Le Breton and Sen [11] and Serizawa [14] are those having a closer relationship to the present work, but only cover the case where the range is a Cartesian product. Our proofs cannot rely on previous results, because the non-cartesian form of the range raises new difficulties. Yet, some steps follow lines similar to those in Barberà and Peleg [1] and Barberà, Sonnenschein and Zhou [3].

The paper is organized as follows. After notation and definitions, Section 2 presents the notion of a generalized median voter scheme. Section 3 provides the characterization of generalized median voter schemes leading to feasible outcomes when individual votes respect feasibility. Section 4 then shows that generalized median voter schemes are in fact the only strategy-proof mechanisms in our setting. The aim of Section 5 is to show that the Gibbard–Satterthwaite Theorem is an immediate consequence of our results. Finally, an Appendix at the end of the paper contains some of the proofs.

2. PRELIMINARIES: GENERALIZED MEDIAN VOTER SCHEMES

Agents are the elements of a finite set $N=\{1, 2, ..., n\}$. We assume that n is at least 2.

Alternatives are K-tuples of integer numbers. For integers $a, b \in \mathbb{Z}$, with $a<b$, we will denote the *integer interval* $[a, b]=\{a, a+1, ..., b\}$. A K-dimensional *box* B is a Cartesian product of K integer intervals: $B=\prod_{k=1}^{K} B_k$, where $B_k=[a_k, b_k]$ and $a_k<b_k$. A *subbox* of B is any box A contained in B. We endow B with the L_1-norm. That is, for every $\alpha \in B$,

$$\|\alpha\| = \sum_{k=1}^{K} |\alpha_k|.$$

Given $\alpha, \beta \in B$, the *minimal box containing* α *and* β is defined by

$$MB(\alpha, \beta)=\{\gamma \in B \mid \|\alpha-\beta\| = \|\alpha-\gamma\| + \|\gamma-\beta\|\}.$$

Preferences are binary relations on alternatives. Let $\mathcal{U}$ be the set of complete, transitive and asymmetric preferences on B. Preferences in $\mathcal{U}$ are denoted by $P, P', P_i, P'_i, ...$. For $P \in \mathcal{U}$, $A \subseteq B$, we denote the alternative in A most preferred by P as $\tau_A(P)$, and we call it *the top of P on A*. $\tau_B(P) \equiv \tau(P)$ will be called the unconstrained top of P.

Preference profiles are n-tuples of preferences on B, $\mathbf{P} \in \mathscr{U}^n$. Profiles $\mathbf{P} = (P_1, ..., P_n)$ are also represented by $(P_i, \mathbf{P}_{-i})$ when we want to stress the role of i's preference. In particular, $(P_i, \mathbf{P}_{-i})$ and $(P'_i, \mathbf{P}_{-i})$ will stand for two profiles which differ in i's preference and are otherwise identical.

A social choice function on $\hat{\mathscr{P}} \subseteq \mathscr{U}$ is a function $F: \hat{\mathscr{P}}^n \to B$.

DEFINITION 1. The social choice function $F: \hat{\mathscr{P}}^n \to B$ *respects voter's sovereignty* on $A \subseteq B$ if for every $\alpha \in A$ there exists $\mathbf{P} \in \hat{\mathscr{P}}^n$ such that $F(\mathbf{P}) = \alpha$.

The range of a social choice function $F: \hat{\mathscr{P}}^n \to B$, is denoted by R_F. That is, $R_F = \{\alpha \in B \mid \text{there exists } \mathbf{P} = (P_1, ..., P_n) \in \hat{\mathscr{P}}^n \text{ such that } F(\mathbf{P}) = \alpha\}$.

We will often start from social choice functions defined on domains $\hat{\mathscr{P}}^n$ and then consider the restriction of F on subsets of $\hat{\mathscr{P}}^n$. When this domain restriction occurs, we identify the restriction of F with itself.

Social choice functions require each agent to report some preference. A social choice function is strategy-proof if it is always in the best interest of agents to reveal their preferences truthfully. Formally,

DEFINITION 2. A social choice function $F: \hat{\mathscr{P}}^n \to B$ is *manipulable on* $\hat{\mathscr{P}} \subseteq \mathscr{U}$ if there exists $\mathbf{P} = (P_1, ..., P_n) \in \hat{\mathscr{P}}^n$, $i \in N$ and $P'_i \in \hat{\mathscr{P}}$ such that $F(P'_i, \mathbf{P}_{-i})\, P_i F(\mathbf{P})$. A social choice function is *strategy-proof on* $\hat{\mathscr{P}}$ if it is not manipulable on $\hat{\mathscr{P}}$.

The preferences of voters are assumed to be strict and to respect a generalized notion of single-peakedness. The preferences P of any voter must be such that they have a *unique* maximal element $\tau(P)$ on B, and whenever alternative α is on a minimal path from β to $\tau(P)$ (and thus closer than β to the best alternative) then $\alpha P \beta$.

DEFINITION 3. A preference P on a box B is *multidimensional single-peaked with bliss point* $\alpha \in B$ iff $\tau(P) = \alpha$, and $\beta P \gamma$ for all $\beta, \gamma \in B$ $(\beta \neq \gamma)$ satisfying $\|\alpha - \gamma\| = \|\alpha - \beta\| + \|\beta - \gamma\|$.

This restriction coincides with the classical notion of single-peakedness when alternatives are one dimensional and preferences are strict. It constitutes a natural extension of single-peakedness to our framework.

We denote by $\mathscr{P}$ the set of all multidimensional single-peaked preferences on B; $\mathscr{P}(\alpha) = \{P \in \mathscr{P} \mid \tau(P) = \alpha\}$ is the set of all such preferences whose top is α. For $A \subseteq B$, $\mathscr{P}(A)$ is the set of all single-peaked preferences with top on A, that is $\mathscr{P}(A) = \bigcup_{\alpha \in A} \mathscr{P}(\alpha)$. We refer to $\mathscr{P}(A)$ as the set of preferences *closed on* A. Notice that $\mathscr{P}(B) = \mathscr{P}$.

We want to single out a special class of social choice functions having the informationally nice property that they only require agents to reveal what is their preferred alternative on the range of the function.

DEFINITION 4. A social choice function $F: \hat{\mathscr{P}}^n \to B$ is a *voting scheme* on R_F if for every $\mathbf{P}, \mathbf{P}' \in \hat{\mathscr{P}}^n$

$$\tau_{R_F}(P_i) = \tau_{R_F}(P'_i)\ \forall i \in N \Rightarrow F(\mathbf{P}) = F(\mathbf{P}').$$

We will say colloquially that voting schemes have the "tops-only" property. Notice that our definition is relative to the range R_F of the function. Thus, two preferences P, P' on B with the same top $\tau(P) = \tau(P')$ outside of R_F may lead to different choices under F, if their tops on the range $\tau_{R_F}(P)$ and $\tau_{R_F}(P')$ are not the same. Therefore, to any voting scheme $F: \hat{\mathscr{P}}^n \to B$ we can associate a function $f: R_F^n \to R_F$ defined by

$$F(P_1, ..., P_n) = f(\tau_{R_F}(P_1), ..., \tau_{R_F}(P_n))$$

for $(P_1, ..., P_n) \in \hat{\mathscr{P}}^n$.

From now on, when this does not lead to confusion, we shall abuse language and also call such a function a voting scheme. When we say then that the voting scheme $f: R_F^n \to R_F$ is strategy-proof on $\hat{\mathscr{P}}$ we mean that the social choice function $F: \hat{\mathscr{P}}^n \to B$ (uniquely) associated with f is strategy-proof on $\hat{\mathscr{P}}$.

We close this section by defining generalized median voter schemes. One description is based on the notion of left-coalition systems. For each dimension k, each set of voters ℓ and each value $\xi \in [a_k, b_k]$ in its range of definition on that dimension, a left-coalition system $\mathscr{L}_k$ indicates whether or not that set of voters, or coalition, belongs to the system at this given value. If it does ($\ell \in \mathscr{L}(\xi)$), this means that the coalition can guarantee the choice of a value not higher than ξ whenever the vote of all members on that dimension is less than or equal to ξ. For this interpretation to be consistent, some natural conditions are required from left-coalition systems. Any left-coalition system generates a unique social choice function on an integer interval, through the following procedure: agents declare their preferred value in the interval, and the chosen value is the (unique) $\bar{\xi}$ having the property that the set of agents voting for values below or equal to $\bar{\xi}$ belongs to the coalition system, while the set of those voting for values strictly below $\bar{\xi}$ does not belong. A similar description can be provided through the symmetric concept of a right-coalition system.

DEFINITION 5. A *left* (*right*)-*coalition system* on $B_k = [a_k, b_k]$ is a correspondence $\mathscr{C}_k$ that assigns to every $\alpha_k \in B_k$ a collection of non-empty coalitions $\mathscr{C}_k(\alpha_k)$ satisfying the following conditions:

(1) If $c \in \mathscr{C}_k(\alpha_k)$ and $c \subset c'$, then $c' \in \mathscr{C}_k(\alpha_k)$.

(2) If $\beta_k > (<)\, \alpha_k$ and $c \in \mathscr{C}_k(\alpha_k)$, then $c \in \mathscr{C}_k(\beta_k)$, and

(3) $\mathscr{C}_k(b_k) = 2^N - \varnothing$ $(\mathscr{C}_k(a_k) = 2^N - \phi)$.

A family $\mathscr{C}$ of left (right)-coalition systems on B is a collection $\{\mathscr{C}_k\}_{k=1}^{K}$ where each $\mathscr{C}_k$ is a left (right)-coalition system on B_k.

Given a left (right)-coalition system $\mathscr{C}_k$ on B_k we say that $c \in \mathscr{C}_k(\alpha_k)$ is a *minimal* left (right) coalition if for every $i \in c$, $c - \{i\} \notin \mathscr{C}_k(\alpha_k)$.

$\mathscr{L}$ will refer to a family of left-coalition systems and $\mathscr{R}$ to a family of right-coalition systems. Moreover $\ell_k(\alpha_k)$ will denote an element of $\mathscr{L}_k(\alpha_k)$ and $r_k(\alpha_k)$ an element of $\mathscr{R}_k(\alpha_k)$. However, α_k and k will be omitted when no confusion may arise.

Definition 6. Let $\tilde{\alpha} \in R_F^n$ and $\beta_k \in B_k$. Define *the coalition to the left* (*right*) of β_k at $\tilde{\alpha}$ by: $\ell(\tilde{\alpha}, \beta_k) = \{i \mid \tilde{\alpha}_k^i \leqslant \beta_k\}$ $(r(\tilde{\alpha}, \beta_k) = \{i \mid \tilde{\alpha}_k^i \geqslant \beta_k\})$.

Definition 7. Let $\mathscr{L} = \{\mathscr{L}_k\}_{k=1}^{K}$ $(\mathscr{R} = \{\mathscr{R}_k\}_{k=1}^{K})$ be a family of left (right)-coalition systems on B. The voting scheme $f\colon R_F^n \to B$ defined as follows:

$$f(\tilde{\alpha}) = \beta \quad \text{iff} \quad \ell(\tilde{\alpha}, \beta_k) \in \mathscr{L}_k(\beta_k)$$
$$\text{and } \ell(\tilde{\alpha}, \beta_k - 1) \notin \mathscr{L}_k(\beta_k - 1)\ \forall k = 1, ..., K \tag{1}$$
$$(f(\tilde{\alpha}) = \beta \quad \text{iff} \quad r(\tilde{\alpha}, \beta_k) \in \mathscr{R}_k(\beta_k)$$
$$\text{and } r(\tilde{\alpha}, \beta_k + 1) \notin \mathscr{R}_k(\beta_k + 1)\ \forall k = 1, ..., K \qquad (1'))$$

is called a *Generalized Median Voter Scheme* (GMVS) defined by $\mathscr{L}(\mathscr{R})$.

It is clear that either left or right coalition systems can be taken as the primitive concept for the definition of a generalized median voter scheme. Proposition 1 gives us the exact connection between the left and right coalition systems associated with a given generalized median voter scheme. This connection will be used extensively in the sequel.

Proposition 1. *Let $f\colon R_F^n \to R_F$ be a generalized median voter scheme defined by $\mathscr{L}$ and also by $\mathscr{R}$. For every $k = 1, 2, ..., K$ let*

$\mathscr{R}_k^*(\alpha_k) = \{r \in 2^N \mid r \cap \ell \neq \varnothing\ \forall \ell \in \mathscr{L}_k(\alpha_k - 1)\}$ *for every $a_k < \alpha_k \leqslant b_k$, and*

$\mathscr{R}_k^*(a_k) = 2^N - \varnothing.$

Then, for every $k = 1, ..., K$ $\mathscr{R}_k^*(\alpha_k) = \mathscr{R}_k(\alpha_k)$ for every $a_k < \alpha_k \leqslant b_k$.

Proof. See the Appendix.

Before finishing this section we would like to single out the class of neutral and anonymous generalized median voter schemes: *voting by quota.* A generalized median voter scheme f is voting by quota if the left (right) coalition system $\mathscr{L}(\mathscr{R})$ that defines it has the following properties, for every $k = 1, ..., K$:

(1) $\mathscr{L}_k(\alpha_k) = \mathscr{L}_k(\beta_k) = \mathscr{L}_k$ $(\mathscr{R}_k(\alpha_k) = \mathscr{R}_k(\beta_k) = \mathscr{R}_k)$ for every $\alpha_k, \beta_k \in B_k$, and

(2) $S \in \mathscr{L}_k (S \in \mathscr{R}_k)$ if and only if $\sharp S \geqslant Q_k^{\ell}$ $(\sharp S \geqslant Q_k^{r})$.

It is easy to see, by using Proposition 1, that the relationship between the left quotas (Q_k^{ℓ}) and the right quotas (Q_k^{r}) defining a quota system is the following: $Q_k^{\ell} + Q_k^{r} = n + 1$.

3. ADMISSIBLE COALITION STRUCTURES UNDER CONSTRAINTS: A CHARACTERIZATION RESULT

Consider the problem of selecting a specific mechanism to achieve the type of collective decisions under consideration. First, there is the question whether we want to look for any type of procedure, or restrict attention to generalized median voter schemes. Our results in Section 4 will prove that, if we are interested in strategy proofness, we should stick to generalized median voter schemes.

But suppose, now, that we know about some a priori feasibility constraints. Say that, while the box B describes the set of conceivable alternatives, only those in $A \subset B$ are actually feasible outcomes. Now the possible choice for the designer must be reassessed in several directions: the picture changes quite dramatically.

First of all, notice that letting agents vote for unfeasible alternatives would easily lead to recommend unfeasible results, since generalized median voter schemes respect unanimity. Thus, we must require agents to vote for feasible alternatives only. But even then, not any generalized median voter scheme will do. Depending on the shape of the set A, and on the structure of the left (or right) coalition system defining f, it may well be, as Example 1 below illustrates, that f would recommend an unfeasible outcome even if each individual voted for a feasible one.

EXAMPLE 1. Consider the case $K = 2$, $[a_1, b_1] = [0, 1]$, $[a_2, b_2] = [0, 1]$, $B = [0, 1]^2$, and $N = \{1, 2, 3, 4\}$. Let $A = \{(0, 0), (0, 1), (1, 0)\}$ and f be voting by right quota 2 $(Q_1^{r} = Q_2^{r} = 2)$. Then, $f((0, 1), (0, 1), (1, 0), (1, 0)) = (1, 1) \notin A$, even if all arguments in the function belong to A. However, if $N = \{1, 2, 3\}$, then the same rule would always guarantee outcomes in A when all agents vote in A.

When f always chooses a feasible outcome, as long as agents vote for feasible ones, we say that f respects feasibility for A under the aggregation process. A designer who is informed of feasibility constraints should concentrate attention on feasibility respecting mechanisms.

DEFINITION 8. A voting scheme $f: B^n \to B$ *respects feasibility on* $A \subset B$ if $f(\alpha^1, \alpha^2, ..., \alpha^n) \in A$ for every $(\alpha^1, \alpha^2, ..., \alpha^n) \in A^n$.

Notice that any generalized median voter scheme will respect feasibility whenever A is a subbox of B.

Suppose then, that given a feasible set A, we take any generalized median voter scheme f respecting feasibility for A. Can we guarantee that f will be strategy-proof on the domain of all single-peaked preferences? Not quite! Remember that now we are requiring agents to vote for their top on the range, and no longer for their unconditionally best alternative. Generalized median voter schemes are strategy-proof when the top alternative of single-peaked agents is feasible and therefore voting for the unrestricted and the restricted top is the same. Otherwise, there would be room for manipulation, as shown by Example 2.

EXAMPLE 2. Consider again the case $K=2$, $[a_1, b_1]=[a_2, b_2]=[0, 1]$, $B=[0, 1]\times[0, 1]$. Let $\hat{\mathscr{P}}=\mathscr{P}$, $N=\{1, 2, 3\}$, and $A=\{(0, 0), (0, 1), (1, 0)\}$. Suppose that the generalized median voter scheme is voting by right quota 2 (as in Example 1). Let P_1 and P_2 be preferences in $\mathscr{P}$ with $\tau(P_1)=(0, 0)$ and $\tau(P_2)=(0, 1)$. Let P_3 and P'_3 be the following preferences for agent 3: $(1, 1)$ $P_3(1, 0)$ $P_3(0, 1)$ $P_3(0, 0)$ and $(0, 1)$ $P'_3(0, 0)$ $P'_3(1, 1)$ $P'_3(1, 0)$. Notice that $\tau_A(P_3)=(1, 0)$ and $\tau_A(P'_3)=\tau(P'_3)=(0, 1)$. Yet F is not strategy-proof since $(0, 1)=F(P_1, P_2, P'_3)$ $P_3 F(P_1, P_2, P_3)=(0, 0)$. However, F is strategy-proof if we let the set of admissible preferences to be exactly $\mathscr{P}(A)$.

What happens is that single-peakedness does not restrict too much the direction of preferences among alternatives that are not top, unless they lie in rather specific positions. Because of that, the strength of single-peakedness depends on the shape of A. We may even have, as an extreme case, a set $A \subset B$ and a class $\hat{\mathscr{P}}$ of single-peaked preferences on B, such that any conceivable ordering of the elements of A is the restriction on A of some single-peaked preferences in $\hat{\mathscr{P}}$ with top in A.

Because of the above, each of our statements must be qualified with reference to the set of preferences where it applies, which in turn is linked to the set of alternatives that are a priori feasible. Given a set A of feasible alternatives, we focus on the domain $\mathscr{P}(A)=\bar{\mathscr{P}}$. Remember (see Section 2) that this is the set of single-peaked preferences closed on A, that is, the set of all preferences whose unconditional top is an element of A. Clearly, our positive results would still hold if domains were restricted further, but not if we enlarge them to allow for preferences with unfeasible tops. Since it is often the case that we prefer what we can not get, our preference domains are very narrow indeed. This is the major reason why overall conclusions have negative flavor.

When characterizing feasibility preserving voting schemes, we'll look for schemes respecting voter's sovereignty on A. This is very natural in our context, since any reason why a feasible alternative should not be in the range could be taken into account as still another form of unfeasibility.

In order to motivate the form of our results, consider the minimal box B containing A, a point α in the box but not in A, and a set $S \subseteq A$. If we can choose a distribution of agents with tops in $S \subseteq A$ whose joint vote under f would lead to α, then f is not feasibility preserving. We look for a condition implying that such distribution cannot be found for any $S \subseteq A$ and any α in the minimal box B (clearly, outcomes outside B will never emerge). Our characterization comes in two parts. We first present a condition, the intersection property, which f must satisfy for any $\alpha \notin A$, $S \subseteq A$ in order to preserve feasibility. The intersection property guarantees some coordination among the decisions taken on each separate dimension by requiring that certain agents belong simultaneously to different coalitions whose separate power could otherwise lead to unfeasible outcomes. Our second result sharpens the above requirement: for each α, there is one set $\hat{S} \subseteq A$, the crucial set, such that, if the intersection property holds for α and $\hat{S}$, then it also holds for α and any $S \subseteq A$. This implies a considerable simplification in the practical use of the condition. We now proceed formally.

For $\alpha, \beta \in B$, let $M^+(\alpha, \beta) = \{k \in \mathbb{K} \mid \beta_k > \alpha_k\}$ and $M^-(\alpha, \beta) = \{k \in \mathbb{K} \mid \beta_k < \alpha_k\}$ be the sets of dimensions in which the components of β are strictly greater or smaller than those of α, respectively.

Definition 9. Let $A \subseteq B$ and let $f: A^n \to B$ be a generalized median voter scheme defined by $\mathscr{L}$ and $\mathscr{R}$. Let $\alpha \notin A$ and $S \subseteq A$. We say that f has the *Intersection Property for* (α, S) iff for every $r(\alpha_k) \in \mathscr{R}(\alpha_k)$ and $\ell(\alpha_k) \in \mathscr{L}(\alpha_k)$ we have that

$$\bigcap_{\beta \in S} \left[\left(\bigcup_{k \in M^+(\alpha, \beta)} \ell(\alpha_k) \right) \cup \left(\bigcup_{k \in M^-(\alpha, \beta)} r(\alpha_k) \right) \right] \neq \varnothing.$$

If $k \notin \bigcup_{\beta \in S} [M^-(\alpha, \beta) \cup M^+(\alpha, \beta)]$, the right or left coalitions selected for this dimension do not play any role in the condition.

We will say that f has the *Intersection Property* if it has it for every $(\alpha, S) \in (B - A, 2^A)$.

THEOREM 1. *Let $f: A^n \to B$ be a generalized median voter scheme respecting voter's sovereignty on A. Then f preserves feasibility on A if and only if it satisfies the intersection property.*

Proof. See the Appendix.

Notice that, when A is a Cartesian product, any choice of values in each of the dimensions leads to a feasible alternative. Hence, the intersection property should not and does not impose any restriction when ranges are Cartesian. Depending on the structure of the set of alternatives the property becomes a more or less stringent requirement. Section 5 includes a proof of the Gibbard–Satterthwaite Theorem based on the family of conditions on left and right coalitions imposed by the intersection property.

Given a feasible set A, an unfeasible alternative $\alpha \notin A$, and a social choice function f, whether or not the intersection property holds would in principle involve calculations for any subset S of A. Our next result will prove that, in fact, one may restrict attention to a single S, what we call the crucial S for α. From now on, we will often identify, without loss of generality, the vector $\alpha \notin B - A$ with the vector $(0, ..., 0)$.

DEFINITION 10. Let $S = \{v^1, ..., v^T\} \subseteq A$ and let $V = S \cup \{v^{T+1}\} \subseteq A$. We say that v^{T+1} is *redundant for* S if for every generalized median voter scheme f respecting voter's sovereignty on A if $\tilde{\xi} = (\xi^1, ..., \xi^n) \in V^n$ is such that $f(\tilde{\xi}) = 0$ then there exists $\tilde{\gamma} = (\gamma^1, ..., \gamma^n) \in S^n$ such that $f(\tilde{\gamma}) = 0$.

For $v \in \mathbb{R}^K$, define $\sup^+(v) = \{k \in \mathbb{K} \mid v_k > 0\}$ and $\sup^-(v) = \{k \in \mathbb{K} \mid v_k < 0\}$.

DEFINITION 11. $S \subseteq A$ ($\sharp S \geqslant 2$) is called crucial if $[v \in S \Rightarrow v$ is not redundant for $S - \{v\}]$ and $[v \notin S \Rightarrow v$ is redundant for $S]$.

DEFINITION 12. The set A *is a support transformation of* A' if $[v \in A - A' \Rightarrow \exists v' \in A'$ such that $\sup^+(v) = \sup^+(v')$ and $\sup^-(v) = \sup^-(v')]$ and $[v' \in A' - A \Rightarrow \exists v \in A$ such that $\sup^+(v') = \sup^+(v)$ and $\sup^-(v') = \sup^-(v)]$.

THEOREM 2. *For any generalized median voter scheme f and every $\alpha \in B - A$, there is a unique crucial set $\hat{S}$ (up to a support transformation) such that f satisfies the intersection property for any (α, S) if and only if it satisfies it for $(\alpha, \hat{S})$.*

Proof. See the Appendix.

Example 3 below illustrates the main concepts used in the statement of Theorem 1 as well as its scope and usefulness to solve the feasibility question for problems of interest.

EXAMPLE 3. A family of problems will arise from cases like the following: a municipality must choose a mix of projects with different levels of intensity. We can represent the conceivable courses of action as K-tuples of integer values. If the municipality can face the cost of any set of packages and has no further restrictions, all these K-tuples will also represent feasible choices.

Budgetary and political constraints will arise in many cases, though. Suppose that high levels of expenditure on all dimensions would exceed the budget, and/or combinations of low levels would be politically unfeasible.

Let's examine in general this family of problems where $B = \prod_{k=1}^{K} [a_k, b_k]$ for the particular case of voting by quota.

By considering only single-peaked preferences whose unconditional tops are feasible we implicitly assume that higher values for a characteristic imply a higher cost, but that citizens do not necessarily prefer the most expensive projects. In some cases, this assumption may be very restrictive.

Consider first the case of a budget constraint coming either from a restriction of a maximum or a minimum amount to spend. The intersection property is equivalent to $\sum_{k=1}^{K} Q_k^r > (K-1)\, n$ for the case of a maximum, and $\sum_{k=1}^{K} Q_k^\ell > (K-1)\, n$ for the case of a minimum. To see that, suppose that $K=2$. It is immediate to check that for $t = r, \ell$:

$$[Q_1^t + Q_2^t > n]$$

$$\Leftrightarrow [S \cap T \neq \emptyset \text{ for all } S, T \subseteq N \text{ such that } \sharp S \geqslant Q_1^t \text{ and } \sharp T \geqslant Q_2^t].$$

Finally, a simple induction argument on K shows that the intersection property is equivalent to both inequalities.

Now, start with the case where both restrictions are in effect at the same time. Using again the fact that $Q_k^\ell + Q_k^r = n+1$ and the condition on the left quotas one obtains that $Kn - \sum_{k=1}^{K} Q_k^r + K > (K-1)\, n$. This, together with the condition on the right quotas implies

$$K + n > \sum_{k=1}^{K} Q_k^r > (K-1)\, n. \tag{2}$$

Therefore a necessary condition is for K and n to satisfy the following inequality:

$$K + n > (K-1)\, n + 1 \Leftrightarrow 2n - 1 > K(n-1) \Leftrightarrow K < \frac{2n-1}{n-1} \leqslant 3.$$

Thus, K must be smaller or equal than 2, which only leaves $K=2$ as an interesting case. By (2) with $K=2$, we have that $n+2>Q_1^r+Q_2^r>n \Rightarrow Q_1^r+Q_2^r=n+1$, which together with $Q_k^\ell+Q_k^r=n+1$ implies $Q_1^\ell+Q_2^\ell=n+1$. Therefore $Q_1^r=Q_2^\ell$ and $Q_1^\ell=Q_2^r$. Moreover, if we impose total neutrality, i.e. $Q_1^\ell=Q_2^\ell=Q^l$, then n has to be odd and $Q^l=Q^\ell=(n+1)/2$.

The above partial impossibility result is due, in part, to the strong restrictions of anonymity and neutrality imposed by voting by quota. It is possible to construct examples showing that, in general, we can find non-trivial generalized median voter schemes when there are maximum and minimum budget constraints and $K>2$.

4. THE STRUCTURE OF STRATEGY-PROOF SOCIAL CHOICE FUNCTIONS

We have learned how to construct feasibility preserving generalized median voter schemes. Results in this section show that, in fact, these are the only social choice functions guaranteeing strategy-proofness as long as voters are restricted to reveal single-peaked preferences on feasible alternatives.

There is a close formal parallelism between our results and those obtained in Barberà, Gul and Stacchetti [2] for functions with Cartesian range. However, since we now deal with arbitrary ranges, we cannot rely on the same proofs, and we must also qualify our statements with reference to specific classes of preferences. We shall be interested in sets containing all single-peaked preferences with unconstrained top on some superset of the range. In the sequel, $\tilde{\mathscr{P}}$ will stand for a generic subset of $\mathscr{P}$, closed on some set containing R_F. In particular, remember that $\bar{\mathscr{P}}=\mathscr{P}(R_F)$ stands for the set of all single-peaked preferences with unconstrained top on the range R_F.

THEOREM 3. *If $F\colon \tilde{\mathscr{P}}^n \to B$ is a strategy-proof social choice function on $\tilde{\mathscr{P}}$, then F is a generalized median voter scheme on R_F.*

Proof. See the Appendix.

Some salient facts on strategy-proof social choice functions emerge along the proof, and we highlight them here, in the form of Propositions 2 and 3, since they are of independent interest. First of all, strategy-proof social choice functions in our setting can only depend on each agent's preferred alternative on A (Proposition 2). Moreover (Proposition 3), they can be decomposed into separate procedures, one for each dimension (see Le Breton and Sen [11]).

PROPOSITION 2. *If $F: \tilde{\mathscr{P}}^n \to B$ is strategy-proof on $\tilde{\mathscr{P}}$, then F is a voting scheme on R_F (it is tops-only on R_F).*

Proof. See the Appendix.

PROPOSITION 3. *If $f: R_F^n \to B$ is a strategy-proof voting scheme on $\tilde{\mathscr{P}}$, then f is a generalized median voter scheme on R_F.*

Proof. See the Appendix.

The proof of Proposition 3 involves a careful examination of the power of coalitions under strategy-proof social choice functions. We provide the main idea of the proof in what follows.

DEFINITION 13. Given a voting scheme $f: R_F^n \to R_F$, for each $k = 1, 2, ..., K$ and every $\alpha_k < b_k$ $(a_k < \alpha_k)$ we define the *set of strong* [*weak*] *left* (*right*) *coalitions* $\bar{\mathscr{L}}_k^f(\alpha_k)$ $(\bar{\mathscr{R}}_k^f(\alpha_k))$ $[\mathscr{L}_k^f(\alpha_k)$ $(\mathscr{R}_k^f(\alpha_k))]$ induced by f on B_k as follows:

$$\mathscr{L}_k^f(\alpha_k) = \{\ell \in 2^N \mid \exists \tilde{\alpha} \in R_F^n \text{ such that } \tilde{\alpha}_k^i \leqslant \alpha_k \ \forall i \in \ell, \tilde{\alpha}_k^i > \alpha_k \ \forall i \notin \ell \text{ and } (f(\tilde{\alpha}))_k \leqslant \alpha_k\},$$

$$\mathscr{R}_k^f(\alpha_k) = \{\imath \in 2^N \mid \exists \tilde{\alpha} \in R_F^n \text{ such that } \tilde{\alpha}_k^i \geqslant \alpha_k \ \forall i \in \imath, \tilde{\alpha}_k^i < \alpha_k \ \forall i \notin \imath \text{ and } (f(\tilde{\alpha}))_k \leqslant \alpha_k\},$$

$$\bar{\mathscr{L}}_k^f(\alpha_k) = \{\ell \in 2^N \mid \text{if } \tilde{\alpha} \in R_F^n \text{ is such that } \tilde{\alpha}_k^i \leqslant \alpha_k \ \forall i \in \ell, \text{ then } (f(\tilde{\alpha}))_k \leqslant \alpha_k\},$$

$$\bar{\mathscr{R}}_k^f(\alpha_k) = \{\imath \in 2^N \mid \text{if } \tilde{\alpha} \in R_F^n \text{ is such that } \tilde{\alpha}_k^i \geqslant \alpha_k \ \forall i \in \imath, \text{ then } (f(\tilde{\alpha}))_k \geqslant \alpha_k\},$$

and $\mathscr{L}_k^f(b_k) = \bar{\mathscr{L}}_k^f(b_k) = \mathscr{R}_k^f(a_k) = \bar{\mathscr{R}}_k^f(a_k) = 2^N - \varnothing$.

What we have called a left (right)-coalition *system* assigns to every point in a one-dimensional box a collection of coalitions with a particular structure (properties (1), (2) and (3) in Definition 5). This concept is, a priori, independent of any voting scheme. On the other hand, the sets of weak or strong left (right) coalitions associated with an arbitrary voting scheme f need not satisfy conditions (1), (2) and (3), even if they are also collections of coalitions for every point in a one-dimensional box. Our proof of Proposition 3 consists in showing that whenever a voting scheme f is strategy proof on $\tilde{\mathscr{P}}$, the sets of weak and strong left (right) coalitions defined by f coincide and are, in fact, a family of left (right) coalition *systems* (see Lemma 2 in the proof of Proposition 3).

We summarize the joint implications of Theorems 1, 2 and 3 in the following Corollary, which only applies when the tops of preferences belong to the feasible set.

COROLLARY 3. *Let $F: \mathscr{P}(A)^n \to A$ be an onto social choice function. Then F is strategy-proof on $\mathscr{P}(A)$ if and only if it is a generalized median voter scheme satisfying the Intersection Property.*

5. THE GIBBARD–SATTERTHWAITE THEOREM

In this section we show how to prove the Gibbard–Satterthwaite Theorem as an implication of our results. This application is also presented as a proof of the strength of our theorems.

A set of agents, $N = \{1, 2, ..., n\}$, must choose one alternative from a finite set $\mathbb{K} = \{1, 2, ..., K\}$ $(K > 2)$. We can still view this problem as a particular case of voting under constraints. To do that, we identify each alternative as one vector of the K-dimensional euclidean basis, and view the remaining vertices of the K-dimensional hypercube as conceivable but unfeasible alternatives. Having embedded our alternatives in $\mathbb{R}^K$, we now want to argue that the assumption that preferences are single-peaked on the K-dimensional hypercube and have tops on feasible alternatives imposes no restriction on the orderings of the K (feasible) alternatives since none of the vectors of the euclidean base lies on the minimal box between two other vectors of the same base (in fact, the L_1-distance between any two of such vectors is always equal to two). Hence, we can apply our results on constrained voting to what in fact is the case considered by Gibbard–Satterthwaite. We will see that one gets the dictator directly from the following two conditions of the intersection property:

(a) For every $j = 1, 2, ..., K$, every $r_j \in \mathscr{R}_j$, and every $k \neq j$, $r_j \cap r_k \neq \varnothing$ for every $r_k \in \mathscr{R}_k$.

(b) For every $\ell_1 \in \mathscr{L}_1$, $\ell_2 \in \mathscr{L}_2$, ..., and $\ell_K \in \mathscr{L}_K$, $\bigcap_{j=1}^{K} \ell_j \neq \varnothing$.

Let j and k be arbitrary $(j \neq k)$ and fix $r_j \in \mathscr{R}_j$. Condition (a) says that $r_j \cap r_k \neq \varnothing$ for every $r_k \in \mathscr{R}_k$. By Proposition 1 we must have that $r_j \in \mathscr{L}_k$. Therefore $\mathscr{R}_j \subseteq \mathscr{L}_k$ for every j and k $(j \neq k)$. We want to show that the inclusions also hold in the other direction. Suppose not, there exists j and k $(j \neq k)$ and $\ell_k \in \mathscr{L}_k$ such that $\ell_k \notin \mathscr{R}_j$. By Proposition 1 there must exist $\ell_j \in \mathscr{L}_j$ such that $\ell_k \cap \ell_j = \varnothing$, contradicting condition (b). Therefore if $K > 2$ the set of left and right coalitions are the same and coincide across the K dimensions. Notice that if $K = 2$, then $\mathscr{R}_1 = \mathscr{L}_2$ and $\mathscr{L}_1 = \mathscr{R}_2$, which does not imply that the generalized median voter scheme is dictatorial. Let $\mathscr{C} = \mathscr{R}_1 = \cdots = \mathscr{R}_K = \mathscr{L}_1 = \cdots = \mathscr{L}_K$.

Next, we want to show that $\mathscr{C}$ only has one minimal set. Suppose otherwise, $D \neq E$ and $D, E \in \mathscr{C}$. Since conditions (a) and (b) have to be satisfied, we must have that $D \cap E \neq \varnothing$. Let $F = D \cap E$. Since D and E are minimal and different we must have that $F \subset D$ and $F \subset E$. Condition (b) implies that $F \cap D' \neq \emptyset$ for every $D' \in \mathscr{C}$, and therefore, by Proposition 1, we have that $F \in \mathscr{C}$, which is a contradiction to the fact that D and E were minimal sets in $\mathscr{C}$. Thus, there must be a unique minimal set in $\mathscr{C}$. Call it D.

To get the dictator, we must show that $\sharp D = 1$. But this has to be the case, since $i \in D$ implies that $\{i\} \cap D' \neq \emptyset$ for every $D' \in \mathscr{C}$, and therefore $\{i\} \in \mathscr{C}$.

APPENDIX

Proof of Proposition 1. Since there is a bijection between right-coalition systems and generalized median voter schemes, it is sufficient to show that

$$[f(\tilde{\alpha})]_k = \beta_k \Leftrightarrow \imath(\tilde{\alpha}, \beta_k) \in \mathscr{R}_k^*(\beta_k) \qquad \text{and} \qquad \imath(\tilde{\alpha}, \beta_k + 1) \notin \mathscr{R}_k^*(\beta_k + 1).$$

Let $[f(\tilde{\alpha})]_k = \beta_k$. Notice that $\imath(\tilde{\alpha}, \beta_k) = [\ell(\tilde{\alpha}, \beta_k - 1)]^c$ and $\imath(\tilde{\alpha}, \beta_k + 1) = [\ell(\tilde{\alpha}, \beta_k)]^c$. Since f is a generalized median voter scheme we have that $\ell(\tilde{\alpha}, \beta_k) = \{i \mid \tilde{\alpha}_k^i \leqslant \beta_k\} \in \mathscr{L}_k(\beta_k)$ and $\ell(\tilde{\alpha}, \beta_k - 1) = \{i \mid \tilde{\alpha}_k^i \leqslant \beta_k - 1\} \notin \mathscr{L}_k(\beta_k - 1)$. To show that $\imath(\tilde{\alpha}, \beta_k) \in \mathscr{R}_k^*(\beta_k)$ suppose not. Then there would exist $\ell' \in \mathscr{L}_k(\beta_k - 1)$ such that $\imath(\tilde{\alpha}, \beta_k) \cap \ell' = \emptyset$. This would imply that $\ell' \subseteq \ell(\tilde{\alpha}, \beta_k - 1)$ contradicting the fact that $\ell(\tilde{\alpha}, \beta_k - 1) \notin \mathscr{L}_k(\beta_k - 1)$. To show that $\imath(\tilde{\alpha}, \beta_k + 1) \notin \mathscr{R}_k^*(\beta_k + 1)$ we have to exhibit a $\ell' \in \mathscr{L}_k(\beta_k)$ such that $\imath(\tilde{\alpha}, \beta_k + 1) \cap \ell' = \emptyset$, but $\ell(\tilde{\alpha}, \beta_k)$ satisfies this condition since its complementary set is $\imath(\tilde{\alpha}, \beta_k + 1)$.

To prove the converse, let $\imath(\tilde{\alpha}, \beta_k) \in \mathscr{R}_k^*(\beta_k)$ and $\imath(\tilde{\alpha}, \beta_k + 1) \notin \mathscr{R}_k^*(\beta_k + 1)$. For every $\ell' \in \mathscr{L}_k(\beta_k - 1)$ we have that $\imath(\tilde{\alpha}, \beta_k) \cap \ell' \neq \emptyset$, but $\imath(\tilde{\alpha}, \beta_k) \cap \ell(\tilde{\alpha}, \beta_k - 1) = \emptyset$ and therefore $\ell(\tilde{\alpha}, \beta_k - 1) \notin \mathscr{L}_k(\beta_k - 1)$. There exists $\ell' \in \mathscr{L}_k(\beta_k)$ such that $\imath(\tilde{\alpha}, \beta_k + 1) \cap \ell' = \emptyset$ implying that $\ell' \subseteq \ell(\tilde{\alpha}, \beta_k) \in \mathscr{L}_k(\beta_k)$. Since f is a generalized median voter scheme we must have that $[f(\tilde{\alpha})]_k = \beta_k$. Q.E.D.

LEMMA 1. *Let $T = \{v^1, v^2, ..., v^T\}$ be a subset of $\mathbb{R}^K$ such that for every $1 \leqslant t \leqslant T$ and every $1 \leqslant k \leqslant K$, v_k^t is equal to either* 1, 0 *or* -1. *Assume there exists $1 \leqslant t \leqslant T$ such that $v^t = (-1, 0, ..., 0)$. Let $\{(A_k^t, B_k^t)_{k=1}^K\}_{t=1}^T$ be a family of subsets of 2^N. Assume*:

(i) *For every $1 \leqslant k \leqslant K$: $v_k^1 = v_k^2 = \cdots = v_k^T$ iff $v_k^t = 0$ for every $1 \leqslant t \leqslant T$.*

(ii) *For every $1 \leqslant t \leqslant T$ and every $1 \leqslant k \leqslant K$: $A_k^t \neq \emptyset$ iff $v_k^t = 1$ and $B_k^t \neq \emptyset$ iff $v_k^t = -1$.*

If $\bigcap_{t=1}^T [(\bigcup_{k=1}^K A_k^t) \cup (\bigcup_{k=1}^K B_k^t)] = \emptyset$ then there exist $\xi^1, ..., \xi^n \in T$ such that for every $1 \leqslant k \leqslant K$ $\{i \in N \mid \xi_k^i \geqslant 0\} \supseteq \bigcup_{t=1}^T B_k^t$ and $\{i \in N \mid \xi_k^i \leqslant 0\} \supseteq \bigcup_{t=1}^T A_k^t$.

Proof. By induction on T. Let $T=2$ and assume that the hypothesis are true for v^1, v^2, and $(A_k^1, B_k^1, A_k^2, B_k^2)_{k=1}^K$. Assume that $v^1, v^2 \neq 0$, otherwise the result is trivial. Define $\xi^1, ..., \xi^n \in \{v^1, v^2\}$ as follows: $\xi^i = v^1$ for every

$$i \in \left(\bigcup_{k=1}^K A_k^2\right) \cup \left(\bigcup_{k=1}^K B_k^2\right)$$

and $\xi^i = v^2$ for every

$$i \in \left[\left(\bigcup_{k=1}^K A_k^1\right) \cup \left(\bigcup_{k=1}^K B_k^1\right)\right]^c;$$

this can be done since the intersection of these two sets of unions is disjoint by hypothesis. Let $1 \leqslant m \leqslant K$ be arbitrary, and assume that $v_m^1 = 1$ (the argument for the other two cases is similar). By construction of the ξ^i's,

$$\{i \in N \mid \xi_m^i \geqslant 0\} \supseteq \left(\bigcup_{k=1}^K A_k^2\right) \cup \left(\bigcup_{k=1}^K B_k^2\right) \supseteq B_m^2 = B_m^1 \cup B_m^2,$$

since $B_m^1 = \varnothing$ by hypothesis, and

$$\{i \in N \mid \xi_m^i \leqslant 0\} \supseteq \left(\bigcup_{k=1}^K A_k^1\right) \cup \left(\bigcup_{k=1}^K B_k^1\right) \supseteq A_m^1 = A_m^1 \cup A_m^2,$$

since $B_m^2 = \varnothing$ by hypothesis.

Assume it is true for T, and $\bigcap_{t=1}^{T+1} [(\bigcup_{k=1}^K A_k^t) \cup (\bigcup_{k=1}^K B_k^t)] = \varnothing$. Without loss of generality assume that $v^{T+1} = (-1, 0, ..., 0)$. Let

$$C = \bigcap_{t=1}^T \left[\left(\bigcup_{k=1}^K A_k^t\right) \cup \left(\bigcup_{k=1}^K B_k^t\right)\right]$$

and define $\bar{A}_k^t = A_k^t - C$ for every $t = 1, ..., T$ and every $k = 1, ..., K$ such that $A_k^t \neq \varnothing$, $\bar{B}_k^t = B_k^t - C$ for every $t = 1, ..., T$ and every $k = 1, ..., K$ such that $B_k^t \neq \varnothing$ except $\bar{B}_1^1$ that we define as

$$\bar{B}_1^1 = [B_k^1 - C] \cup \left[\bigcup_{t=1}^T \left[\left(\bigcup_{k=1}^K A_k^t\right) \cup \left(\bigcup_{k=1}^K B_k^t\right)\right]\right]^c.$$

By construction

$$\bigcap_{t=1}^T \left[\left(\bigcup_{k=1}^K \bar{A}_k^t\right) \cup \left(\bigcup_{k=1}^K \bar{B}_k^t\right)\right] = \varnothing.$$

Our induction hypothesis is that there exist $\xi^1 \in \{v^1, ..., v^T\}$ for every $i \in N - C$ such that for every $1 \leqslant k \leqslant K$,

$$\{i \in N - C \mid \xi^i_k \geqslant 0\} \supseteq \bigcup_{t=1}^{T} \bar{B}^t_k \qquad \text{and} \qquad \{i \in N - C \mid \xi^i_k \leqslant 0\} \supseteq \bigcup_{t=1}^{T} \bar{A}^t_k.$$

For every $i \in C$ define $\xi^i = v^{T+1}$. Then for every $k > 1$,

$$\{i \in N \mid \xi^i_k \geqslant 0\} \supseteq \bigcup_{t=1}^{T} \bar{B}^t_k \cup C = \bigcup_{t=1}^{T+1} \bar{B}^t_k$$

since $B^{T+1}_k = \varnothing$,

$$\{i \in N \mid \xi^i_1 \geqslant 0\} \supseteq \bigcup_{t=1}^{T} \bar{B}^t_k = \bigcup_{t=1}^{T+1} \bar{B}^t_k$$

since $B^{T+1}_k \cap C = \varnothing$ and by definition of $\bar{B}^1_1$, and for every $1 \leqslant k \leqslant K$,

$$\{i \in N \mid \xi^i_k \leqslant 0\} \supseteq \bigcup_{t=1}^{T} \bar{A}^t_k \cup C = \bigcup_{t=1}^{T+1} A^t_k$$

since $A^{T+1}_k = \varnothing$, which proves the lemma. Q.E.D.

Proof of Theorem 1. *Sufficiency*: Voter's sovereignty on A implies that $f(A^n) \supseteq A$. Therefore we must show that $f(A^n) \subseteq A$. Assume f has the Intersection Property for every (α, S), and suppose f does not preserve feasibility; that is, there exists $\alpha \notin A$ and $\tilde{\xi} \in A^n$ such that $f(\tilde{\xi}) = \alpha$. Define $S = \{\xi^1, ..., \xi^n\}$ and the sets $C^+_k = \{i \in N \mid \xi^i_k \geqslant \alpha_k\}$ for $k \in \bigcup_{i \in N} M^-(\alpha, \xi^i)$ and $C^-_k = \{i \in N \mid \xi^i_k \leqslant \alpha_k\}$ for $k \in \bigcup_{i \in N} M^+(\alpha, \xi^i)$. Since f is a generalized median voter scheme we know that $C^+_k \in \mathscr{R}_k(\alpha_k)$ and $C^-_k \in \mathscr{L}_k(\alpha_k)$. By hypothesis

$$\bigcap_{i \in N} \left[\left(\bigcup_{k \in M^+(\alpha, \xi^i)} C^-_k \right) \cup \left(\bigcup_{k \in M^-(\alpha, \xi^i)} C^+_k \right) \right] \neq \varnothing.$$

Therefore, by the distributive law between unions and intersections, $\bigcap_{i \in N} C(\xi^i) \neq \varnothing$ for a selection of $C(\xi^i)$ satisfying

$$C(\xi^i) = \begin{cases} C^-_{k_i} & \text{if} \quad k_i \in M^+(\alpha, \xi^i) \\ C^+_{k_i} & \text{if} \quad k_i \in M^-(\alpha, \xi^i) \end{cases}.$$

For every $i \in N$, we have that $M^+(\alpha, \xi^i) \neq \varnothing$ or $M^-(\alpha, \xi^i) \neq \varnothing$, otherwise $\xi^i = \alpha$.

Let $j \in \bigcap_{i \in N} C(\xi^i)$. For every $i \in N$, $C(\xi^i)$ is equal to either:

(1) $C^+_{k_i}$ if $k_i \in M^-(\alpha, \xi^i)$ which means that $\xi^i_{k_i} < \alpha_{k_i}$; but $j \in C^+_{k_i}$ implies $\xi^j_{k_i} \geqslant \alpha_{k_i}$, therefore $\xi^j \neq \xi^i$, or

(2) $C^-_{k_i}$ if $k_i \in M^+(\alpha, \xi^i)$ which means that $\xi^i_{k_i} > \alpha_{k_i}$; but $j \in C^-_{k_i}$ implies $\xi^j_{k_i} \leqslant \alpha_{k_i}$, therefore $\xi^j \neq \xi^i$.

(1) and (2) constitute a contradiction with the fact that there exists $\xi \in S$ such that $\xi = \xi^j$.

Necessity: We will show the contrapositive. Suppose f does not satisfy the intersection property. In order to apply the preceding lemma, we make the following transformation: for every $1 \leqslant t \leqslant T$, define v^t as follows

$$v^t_k = \begin{cases} 1 & \text{if } \quad \beta^t_k > \alpha_k \\ 0 & \text{if } \quad \beta^t_k = \alpha_k \\ -1 & \text{if } \quad \beta^t_k < \alpha_k. \end{cases}$$

Define $\xi(\alpha) = (0, ..., 0)$ and $V = \{v^1, ..., v^T\}$. For every $1 \leqslant t \leqslant T$ let

$$A^t_k\ (B^t_k) = \begin{cases} \ell_k(\alpha_k)\ (\imath_k(\alpha_k)) & \text{if } \quad k \in M^-(\alpha, \beta^t)\ (k \in M^+(\alpha, \beta^t)) \\ \varnothing & \text{if } \quad k \notin M^-(\alpha, \beta^t)\ (k \notin M^+(\alpha, \beta^t)). \end{cases}$$

Thus, by the previous lemma there exist $\xi^1, ..., \xi^n \in V$ such that for every $1 \leqslant k \leqslant K$, $\{i \in N \mid \xi^i_k \geqslant 0\} \supseteq \bigcup_{t=1}^T B^t_k = B_k$, and $\{i \in N \mid \xi^i_k \leqslant 0\} \supseteq \bigcup_{t=1}^T A^t_k = A_k$. This means that there exist $\beta^1, ..., \beta^n \in S$ such that for every $1 \leqslant k \leqslant K$, $\{i \in N \mid \xi^i_k \geqslant 0\} = \{i \in N \mid \beta^i_k \geqslant \alpha_k\} \supseteq B_k = \imath_k(\alpha_k)$ and $\{i \in N \mid \xi^i_k \leqslant 0\} = \{i \in N \mid \beta^i_k \leqslant \alpha_k\} \supseteq A_k = \ell_k(\alpha_k)$. Since f is a generalized median voter scheme we have that for every $1 \leqslant k \leqslant K$, $f_k(\beta^1, ..., \beta^n) = \alpha_k$, which implies that there exists $\tilde{\beta} \in A^n$ such that $f(\tilde{\beta}) = \alpha \notin A$. Q.E.D.

Theorem 3 follows from Propositions 2 and 3. Although it would be natural to first prove that we should restrict attention to voting schemes (Proposition 2) and then show that strategy-proof voting schemes must be generalized median voter schemes (Proposition 3), it is convenient to reverse this order. This is because the proof of Proposition 2 involves an induction argument on the number of agents, while Proposition 3 can be proved directly for any n, and its conclusion is used for the induction argument.

LEMMA 2. *For every $\tilde{\alpha} \in R^n_F$, every $k = 1, ..., K$ and every $\beta_k \in B_k$*

$$\ell(\tilde{\alpha}, \beta_k) \in \mathscr{L}^f_k(\beta_k) \Leftrightarrow [f(\tilde{\alpha})]_k \leqslant \beta_k.$$

Proof. Sufficiency is obvious by the definition of $\ell(\tilde{\alpha}, \beta_k)$ and $\mathscr{L}^f_k(\beta_k)$. To show necessity, let $\ell(\tilde{\alpha}, \beta_k) \in \mathscr{L}^f_k(\beta_k)$. Then $\exists \tilde{\gamma} \in R^n_F$ such that $\tilde{\gamma}^i_k \leqslant \beta_k$ $\forall i \in \ell(\tilde{\alpha}, \beta_k)$, $\tilde{\gamma}^i_k > \beta_k$ $\forall i \notin \ell(\tilde{\alpha}, \beta_k)$ and $[f(\tilde{\gamma})]_k \leqslant \beta_k$. We will show that $[f(\tilde{\alpha})]_k \leqslant \beta_k$. We consider the sequence $(\tilde{\gamma}^{-i} \mid \tilde{\alpha}^i)$ for $i \in \ell(\tilde{\alpha}, \beta_k)$ and prove that $[f(\tilde{\gamma}^{-i} \mid \tilde{\alpha}^i)]_k \leqslant \beta_k$. Since f is strategy proof, $f(\tilde{\gamma}^{-i} \mid \tilde{\alpha}^i) \in MB(\tilde{\alpha}^i, f(\tilde{\gamma}))$. Otherwise take $P_i \in \tilde{\mathscr{P}}$ such that: $\tilde{\alpha}^i = \tau(P_i)$ and $[\xi \in MB(\tilde{\alpha}^i, f(\tilde{\gamma})) \Rightarrow \xi P_i f(\tilde{\gamma}^{-i} \mid \tilde{\alpha}^i)]$. Then player i can manipulate at $(\tilde{\gamma}^{-i} \mid \tilde{\alpha}^i)$, since $f(\tilde{\gamma})$ $P_i f(\tilde{\gamma}^{-i} \mid \tilde{\alpha}^i)$. Notice that $\tilde{\alpha}^i_k \leqslant \beta_k$ and $[f(\tilde{\gamma})]_k \leqslant \beta_k$, which implies that $[f(\tilde{\gamma}^{-i} \mid \tilde{\alpha}^i)]_k \leqslant \beta_k$. By the same type of reasoning, and changing sequentially $\tilde{\gamma}^i$ by $\tilde{\alpha}^i$ for $i \notin \ell(\tilde{\alpha}, \beta_k)$ one obtains that $[f(\tilde{\alpha})]_k \leqslant \beta_k$. Q.E.D.

Proof of Proposition 3. To show that f is a generalized median voter scheme defined by $\mathscr{L}^f$ one has just to establish the equivalence

$$f(\tilde{\alpha}) = \beta \qquad \text{iff} \quad \ell(\tilde{\alpha}, \beta_k) \in \mathscr{L}^f(\beta_k)$$
$$\text{and } \ell(\tilde{\alpha}, \beta_k - 1) \notin \mathscr{L}^f(\beta_k - 1) \quad \forall k = 1, ..., K,$$

since $\mathscr{L}^f$ is a family of left-coalition systems.

Let $f(\tilde{\alpha}) = \beta$. Consider $\ell(\tilde{\alpha}, \beta_k)$, which by construction belongs to $\mathscr{L}^f_k(\beta_k)$. Assume that $\ell(\tilde{\alpha}, \beta_k - 1) \in \mathscr{L}^f_k(\beta_k - 1)$. Then $[f(\tilde{\alpha})]_k < \beta_k$ by Lemma 2, which is a contradiction.

To show the other direction in the equivalence, notice that Lemma 2 establishes that $[f(\tilde{\alpha})]_k \leqslant \beta_k$ for every $k = 1, 2, ..., K$. It suffices to show that $[f(\tilde{\alpha})]_k \geqslant \beta_k$. This is because $\ell(\tilde{\alpha}, \beta_k - 1) \notin \mathscr{L}^f_k(\beta_k - 1) \Rightarrow [f(\tilde{\alpha})]_k > \beta_k - 1 \Rightarrow [f(\tilde{\alpha})]_k \geqslant \beta_k$. Q.E.D.

Lemma 2 used in the proof of Proposition 3 implies that the set of weak and strong left (right) coalitions at β_k are indeed the same.

Our next definition will be useful in the proof of Proposition 2, which follows a line of reasoning initiated in Barberà and Peleg [1].

DEFINITION 14. For a social choice function $F: \tilde{\mathscr{P}}^n \to B$, the *set of options*, given $P_i \in \tilde{\mathscr{P}}$, is defined by

$$o^F_{-i}(P_i) = \{\alpha \in B \mid \exists \mathbf{P}_{-i} \in \tilde{\mathscr{P}}^{n-1} \text{ such that } F(P_i, \mathbf{P}_{-i}) = \alpha\}.$$

Proof of Proposition 2. By induction on n, the number of agents.

Step 1 ($n = 2$): If $F: \tilde{\mathscr{P}}^2 \to B$ is strategy-proof on $\tilde{\mathscr{P}}$, then F is a voting scheme on R_F.

Proof of Step 1. By repeatedly using the fact that for two person social choice functions, strategy proofness implies that for every profile (P_1, P_2), $F(P_1, P_2)$ is the P_2-maximal element on $o^F_2(P_1)$ and also the P_1-maximal element on $o^F_1(P_2)$, it is easy to show that strategy proofness of F implies that if $o_2^F(P_1)=o_2^F(P_1')$ then $F(P_1, P_2)=F(P_1', P_2)$. Therefore, following similar techniques first introduced by Barberà and Peleg [1], we have that for all P_1 and P_1' in $\tilde{\mathscr{P}}$

$$\tau_{R_F}(P_1)=\tau_{R_F}(P_1') \qquad \text{implies} \quad o_2^F(P_1)=o_2^F(P_1'). \tag{3}$$

Step 2 ($n>2$): We want to show that if $(P_1, ..., P_n)\in\tilde{\mathscr{P}}^n$ and $(P_1', ..., P_n')\in\tilde{\mathscr{P}}^n$ are such that $\tau_{R_F}(P_i)=\tau_{R_F}(P_i')$ for every $i\in N$, then $F(P_1, ..., P_n)=F(P_1', ..., P_n')$. The result is obtained by repeated use of the following lemma.

LEMMA: *If $P_1, P_1'\in\tilde{\mathscr{P}}$ are such that $\tau_{R_F}(P_1)=\tau_{R_F}(P_1')$, then for every $(P_2, ..., P_n)\in\tilde{\mathscr{P}}^{n-1}$ one has that $F(P_1', P_2, ..., P_n)=F(P_1, P_2, ..., P_n)$.*

Proof of lemma.

Claim 1. $\tau_{R_F}(P_1)=\tau_{R_F}(P_1')\Rightarrow o_{-1}^F(P_1)=o_{-1}^F(P_1')$.

Proof of claim 1. Consider the function $G\colon\tilde{\mathscr{P}}^2\to B$ defined by $G(P_1, P_2)=F(P_1, P_2, ..., P_2)$ for every $(P_1, P_2)\in\tilde{\mathscr{P}}^2$. It is easy to show that $R_F=R_G$ and that F being strategy proof implies that G is also strategy proof. Therefore, we can use (see (3) above) that $o_2^G(P_1)=o_2^G(P_1')$ for every P_1 and P_1' with the same top on R_F. Hence, it is sufficient to show that $o_2^G(P_1)=o_{-1}^F(P_1)$. Clearly $o_2^G(P_1)\subseteq o_{-1}^F(P_1)$. To show the other inclusion suppose that there exists α such that $F(P_1, P_2, ..., P_n)=\alpha\notin o_2^G(P_1)$. For $P\in\tilde{\mathscr{P}}(\alpha)$, strategy proofness of F implies that $F(P_1, P, P_3, ..., P_n)=\alpha$ because otherwise, agent 2 could manipulate at this profile. One gets the result by replacing sequentially P for $P_i (i>2)$.

Claim 2. $F(P_1', P_2, ..., P_n)=F(P_1, P_2, ..., P_n)$.

Proof of claim 2. By induction on n. If $n=2$, strategy proofness implies that agent 2 maximizes over the option set left by agent 1, and therefore if the sets of options are the same, F has to take the same value.

Now, suppose the induction hypothesis holds for $n-1$, but not for n. Then, there exist $(P_2, ..., P_n)\in\tilde{\mathscr{P}}^{n-1}$ such that

$$F(P_1', P_2, ..., P_n)=\alpha'\neq\alpha=F(P_1, P_2, ..., P_n). \tag{4}$$

Define, for all $P_1\in\tilde{\mathscr{P}}$, $F_{P_1}\colon\tilde{\mathscr{P}}^{n-1}\to B$ by $F_{P_1}(P_2, ..., P_n)=F(P_1, P_2, ..., P_n)$. Denote F_{P_1} and $F_{P_1'}$ by G and G' respectively. Since F is strategy proof on $\tilde{\mathscr{P}}$, G and G' are strategy-proof on $\tilde{\mathscr{P}}$. Clearly $R_G=R_{G'}$, since $R_G=o_{-1}^F(P_1)=o_{-1}^F(P_1')=R_{G'}$. Since $\tilde{\mathscr{P}}\supseteq\mathscr{P}(R_F)$ and $R_F\supseteq R_G$ we have that $\mathscr{P}(R_G)\subseteq\mathscr{P}(R_F)\subseteq\tilde{\mathscr{P}}$. By the induction hypothesis G and G' are tops only on R_G. Therefore G and G' can be considered functions from $(R_G)^{n-1}$ to R_G, which are strategy-proof on $\tilde{\mathscr{P}}\supseteq\mathscr{P}(R_G)$. Our contradiction hypothesis implies that there exist $\alpha^2, ..., \alpha^n\in R_G$, $\alpha^i=\tau_{R_G}(P_i)$ for $i=2, ..., n$, such that

$G(\alpha^2, ..., \alpha^n) = \alpha \neq \alpha' = G'(\alpha^2, ..., \alpha^n)$. Proposition 3 implies that G and G' are generalized median voter schemes. Since $\alpha \neq \alpha'$, we can assume that there exists $K \geqslant k \geqslant 1$ such that $\alpha_k > \alpha'_k$. Define $\ell = \{i > 1 \mid \alpha^i_k \leqslant \alpha'_k\}$ and $\imath = \{i > 1 \mid \alpha^i_k \geqslant \alpha_k\}$. Since the kth component of G and G' are different one must have that $\ell \in \mathscr{L}'_k(\alpha'_k)$ and $\ell \notin \mathscr{L}_k(\alpha'_k)$, $\imath \in \mathscr{R}_k(\alpha_k)$ and $\imath \notin \mathscr{R}'_k(\alpha_k)$, where $\mathscr{L}'_k$ and $\mathscr{L}_k$ ($\mathscr{R}'_k$ and $\mathscr{R}_k$) are the left (right) coalitions associated to G' and G, respectively. Let $\beta = \tau_{R_F}(P_1) = \tau_{R_F}(P'_1)$. Either $\alpha'_k < \beta_k$ or $\beta_k < \alpha_k$. We are going to obtain a contradiction for the first case using the left coalition system; the second case is similar using the right coalition system. Let $\eta = (\eta^2, ..., \eta^n)$ where $\eta^i = \alpha'$ if $i \in \ell$ and $\eta^i = \beta$ if $i \notin \ell$. Then, since G is a generalized median voter scheme, $G(\eta) \in MB(\alpha', \beta)$, $[G(\eta)]_k > \alpha'_k$ and $[G'(\eta)]_k = \alpha'_k$. We are going to obtain, first, a contradiction for the case where $\beta = \tau(P_1) = \tau(P'_1) \in R_F$, and then, using this fact, obtain a contradiction for the general case. Assume that $\beta = \tau(P_1) = \tau(P'_1) \in R_F$. If $G'(\eta) = \alpha'$ there exist $\bar{P}_i \in \mathscr{P}(\eta^i)$ for $i > 1$ such that $F(P_1, \bar{P}_2, ..., \bar{P}_n)\, P'_1\, F(P'_1, \bar{P}_2, ..., \bar{P}_n)$ which contradicts strategy-proofness of F. If $G'(\eta) \neq \alpha'$ replace in the above argument α' by $G'(\eta)$. Since $G'(\eta)$ was an element of $MB(\alpha', \beta)$ iterating this argument one gets $\bar{\eta} = (\bar{\eta}^2, ..., \bar{\eta}^n) \in (R_G)^{n-1}$ where $\bar{\eta}^i = \gamma$ if $i \in \ell$, $\bar{\eta}^i = \beta$ if $i \notin \ell$, $\gamma_k = \alpha'_k$, $G'(\bar{\eta}) = \gamma$ and $G(\bar{\eta}) \in MB(\gamma, \beta) - \{\gamma\}$. Then, there exist $\hat{P}_i \in \mathscr{P}(\bar{\eta}^i)$ for $i > 1$ such that $G(\bar{\eta}) = F(P_1, \hat{P}_2, ..., \hat{P}_n)\, P'_1\, F(P'_1, \hat{P}_2, ..., \hat{P}_n) = G'(\bar{\eta})$ which contradicts strategy-proofness of F. Assume now that $\beta = \tau_{R_F}(P_1) = \tau_{R_F}(P'_1) \in R_F$. By the same argument just used above we can identify a $\bar{\eta} = (\bar{\eta}^2, ..., \bar{\eta}^n) \in (R_G)^{n-1}$ where $\bar{\eta}^i = \gamma$ if $i \in \ell$, $\bar{\eta}^i = \beta$ if $i \notin \ell$, $\gamma_k = \alpha'_k$, $G'(\bar{\eta}) = \gamma$ and $G(\bar{\eta}) \in MB(\gamma, \beta) - \{\gamma\}$. Consider any $\hat{P}_1, \hat{P}'_1 \in \mathscr{P}(\beta)$ and let $\hat{G}$ and $\hat{G}'$ be the associated generalized median voter schemes on $\tilde{\mathscr{P}}^{n-1}$ once $\hat{P}_1$ and $\hat{P}'_1$ are fixed. We have just seen that in this case, $\hat{G}$ and $\hat{G}'$ coincide, hence $\hat{G}(\bar{\eta}) = \hat{G}'(\bar{\eta}) \in MB(G'(\bar{\eta}), \beta)$. We must have that $G(\bar{\eta}) \notin MB(\hat{G}(\bar{\eta}), \beta)$, otherwise it would contradict strategy proofness of F, since $F(P_1, \hat{P}_2, ..., \hat{P}_n)\, \hat{P}_1\, F(\hat{P}_1, \hat{P}_2, ..., \hat{P}_n)$ for $\hat{P}_i \in \mathscr{P}(\bar{\eta}^i)$ for $i > 1$. Therefore, there exists $\bar{P}_1 \in \mathscr{P}(\beta)$ (and its associated $\bar{G}$) such that $\beta \bar{P}_1 G(\bar{\eta})\, \bar{P}_1\, \hat{G}(\bar{\eta})$. Now, $\hat{G}(\bar{\eta}) = \hat{G}'(\bar{\eta}) = \bar{G}(\bar{\eta})$ and therefore we obtain the contradiction of strategy proofness of F since $G(\bar{\eta}) = F(P_1, \hat{P}_2, ..., \hat{P}_n)\, \bar{P}_1\, F(\bar{P}_1, \hat{P}_2, ..., \hat{P}_n) = \bar{G}(\bar{\eta})$ for $\hat{P}_i \in \mathscr{P}(\bar{\eta}^i)$ for $i > 1$. Q.E.D.

Theorem 2 follows trivially from Propositions 5, 6 and 7 below.

PROPOSITION 5. *Let v^{T+1} and S be given. v^{T+1} is redundant for S if and only if there exists $v^t \in S$ such that* $\sup^+(v^t) \subseteq \sup^+(v^{T+1})$ *and* $\sup^-(v^t) \subseteq \sup^-(v^{T+1})$.

Proof. To prove sufficiency, let v^{T+1} and S be given. Assume, without loss of generality, that v^1 is such that $\sup^+(v^1) \subseteq \sup^+(v^{T+1})$ and $\sup^-(v^1) \subseteq \sup^-(v^{T+1})$. Let $\tilde{\xi} \in V^n$ and assume that $f(\tilde{\xi}) = 0$. Then by the intersection property, there exist $c_1 \in \mathscr{L}_1(0) \cup \mathscr{R}_1(0)$, $c_{k_2} \in \mathscr{L}_{k_2}(0) \cup \mathscr{R}_{k_2}(0)$, ..., $c_{k_T} \in \mathscr{L}_{k_T}(0) \cup \mathscr{R}_{k_T}(0)$, $c_{k_{T+1}} \in \mathscr{L}_{k_{T+1}}(0) \cup \mathscr{R}_{k_{T+1}}(0)$ such that $c_1 \cap (\bigcap_{t=2}^{T+1} c_{k_t}) = \varnothing$. Moreover, $c_1 = c_{k_{T+1}}$ since $\sup^+(v^1) \subseteq \sup^+(v^{T+1})$ and $\sup^-(v^1) \subseteq \sup^-(v^{T+1})$. Therefore, the empty intersection can be expressed as $c_1 \cap (\bigcap_{t=2}^{T} c_{k_t}) = \varnothing$. Thus, there exist $\tilde{\gamma} = (\gamma^1, ..., \gamma^n) \in S^n$ such that $f(\tilde{\gamma}) = 0$.

To show necessity, we will prove the contrapositive. Assume that for every $1 \leqslant t \leqslant T$, $\sup^+(v^t) - \sup^+(v^{T+1}) \neq \varnothing$ or $\sup^-(v^t) - \sup^-(v^{T+1}) \neq \varnothing$. We have to show that there exists a generalized median voter scheme satisfying voter's sovereignty such that $f(\tilde{\xi}) = 0$ for a $\tilde{\xi} \in V^n$ and $f(\tilde{\gamma}) \neq 0$ for every $\tilde{\gamma} \in S^n$. Without loss of generality assume that $T + 1 = K$ and for every $1 \leqslant t = k \leqslant K$, $\sup^+(v^k) = k$ and $\sup^-(v^k) = \varnothing$. Consider any generalized median voter scheme f defined by $\mathscr{L}$ satisfying the following property: $\ell_1(0) = \cdots = \ell_{K-1}(0) = \{1\}$ and $\ell_K(0) = \{2\}$. Then, since $\bigcap_{k=1}^{K-1} \ell_k(0) = \{1\}$ and $\bigcap_{k=1}^{K} \ell_k(0) = \{1\} \cap \{2\} = \varnothing$ we must have that $f(\tilde{\gamma}) \neq 0$ for every $\tilde{\gamma} \in S^n$ and there exists $\tilde{\xi} \in V^n$ such that $f(\tilde{\xi}) = 0$. Q.E.D.

PROPOSITION 6. *Given $A \subseteq B$ there exists, up to a support transformation, a unique crucial subset of A.*

Proof. Existence is trivial since A is finite. Suppose $\hat{S} \neq S'$ are crucial subsets of A. Without loss of generality assume there exists $v \in \hat{S} - S'$. Since $v \notin S'$ and S' is crucial, v is redundant for S'; therefore there exists $v' \in S'$ such that $\sup^+(v) \subseteq \sup^+(v')$ and $\sup^-(v) \subset \sup^-(v')$. If $v' \in \hat{S}$ then v is redundant for $\hat{S}$ which contradicts that $\hat{S}$ is crucial. If $v' \notin \hat{S}$ then it is redundant for $\hat{S}$; therefore there exists $\hat{v} \in \hat{S}$ such that $\sup^+(v') \subseteq \sup^+(\hat{v})$ and $\sup^-(v') \subset \sup^-(\hat{v})$. Therefore $\sup^+(v) \subseteq \sup^+(\hat{v})$ and $\sup^-(v) \subset \sup^-(\hat{v})$, but $v, \hat{v} \in \hat{S}$ contradicting that $\hat{S}$ is crucial. Q.E.D.

PROPOSITION 7. *Suppose that $\hat{S} \subseteq A$ is crucial and $\alpha \notin A$. Then for every generalized median voter scheme f the following is true: For every $S \subseteq A$, f has the intersection property for (α, S) if and only if f has the intersection property for $(\alpha, \hat{S})$.*

Proof. Let f be a generalized median voter scheme. Necessity is trivial. To show sufficiency, assume that there exists $S \subseteq A$ such that f does not have the intersection property for (α, S). Then f does not also have the intersection property for $(\alpha, S \cup \hat{S})$. Since $\hat{S}$ is crucial, every $v \in S - \hat{S}$ is redundant for $\hat{S}$, therefore, by Proposition 1, f does not have the intersection property for $(\alpha, S \cup \hat{S} - \{v\})$. We can keep subtracting from the union elements in $S - \hat{S}$ until we obtain that f does not have the intersection property for $(\alpha, \hat{S})$. Q.E.D.

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